Intersection numbers with Witten's top Chern class

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Abstract

Witten's top Chern class is a particular cohomology class on the moduli space of Riemann surfaces endowed with r-spin structures. It plays a key role in Witten's conjecture relating to the intersection theory on these moduli spaces.

Our first goal is to compute the integral of Witten's class over the so-called double ramification cycles in genus 1. We obtain a simple closed formula for these integrals.

This allows us, using the methods of [15], to find an algorithm for computing the intersection numbers of the Witten class with powers of the ψ-classes (or tautological classes) over any moduli space of r-spin structures, in short, all numbers involved in Witten's conjecture.

1 Introduction

1.1 Aims and purposes

In 1991 E. Witten formulated two conjectures relating to the intersection theory of moduli spaces of curves [16, 17], motivated by two dimensional gravity.

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The first conjecture involves moduli spaces of stable curves and the tautological 2-cohomology classes on them (also called ψ-classes). The intersection numbers of powers of the ψ-classes can be arranged into a generating series that is claimed to be a solution of the Korteweg – de Vries (or KdV) hierarchy of partial differential equations. This conjecture was first proved by M. Kontsevich in [9]. At present there are several alternative proofs: [12], [10], [7], [8].

The second conjecture is still open, and even giving a precise formulation required joint work by several people (references are given below). It involves a more complicated moduli space, called the *space of r-spin structures*. Apart from the ψ-classes, one considers one more cohomology class, called the *Witten top Chern class*, or just *Witten's class* for shortness. We are interested in the intersection numbers of Witten's class with powers of the ψ-classes. These intersection numbers can, once again, be arranged into a generating series, and this series is claimed to give a solution of the rth Gelfand-Dikii (or the r-KdV) hierarchy.

The precise definition of Witten's top Chern class is rather involved; however it is known to satisfy quite simple factorization rules.

In [15] the first author found an "almost-algorithm" for computing some of the intersection numbers arising in Witten's second conjecture *using only the factorization rules for the Witten class*. More precisely, the factorization rules allow one to express more complicated intersection numbers via simpler ones, until one arrives at unsimplifiable cases. These can be of two types. (i) Integrals of Witten's class (with no ψ-classes) over genus 0 moduli spaces. These numbers are well-known. (ii) Integrals of Witten's class (with no ψ-classes) over some special divisors on genus 1 moduli space. (These divisors have a rather cumbersome name of *double ramification divisors* - see below.) When numbers of second type appeared in the course of computations the algorithm blocked without giving an answer.

The purpose of this note it twofold.

First, compute the integrals of Witten's class over the double ramification divisors in genus 1. It turns out that a simple closed formula exists for these integrals.

Second, complete and give a coherent exposition of the algorithm for computing Witten's intersection numbers. Our computation uses only factorization rules for Witten's class. Therefore, we obtain the following theorem.

Theorem 1 *The intersection numbers of Witten's class with powers of the ψ-classes are entirely determined by (i) genus* 0 *intersection numbers involving no ψ-classes, and (ii) the factorization rules for Witten's class.*

1.2 Main definitions

1.2.1 Moduli spaces.

$\mathcal{M}_{g,n}$ is the moduli space of smooth complex genus g curves with $n \geq 1$ distinct numbered marked points. $\overline{\mathcal{M}}_{g,n}$ is its Deligne-Mumford compactification, in other words, the moduli space of stable curves. Over $\overline{\mathcal{M}}_{g,n}$ we define n holomorphic line bundles $\mathcal{L}_i$. The fiber of $\mathcal{L}_i$ over a point $a \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line to the corresponding stable curve C_a at the ith marked point. The first Chern classes $\psi_i = c_1(\mathcal{L}_i)$ of these line bundles are called the *tautological classes*.

1.2.2 Spaces of r-spin structures.

Chose an integer $r \geq 2$ and pick n integers $a_1, \dots a_n \in \{0, \dots, r-1\}$ in such a way that $2g - 2 - \sum a_i$ is divisible by r. The numbers $a_1, \dots, a_n$ are assigned to the marked points $x_1, \dots, x_n$. On a smooth curve C one can find r^{2g} different line bundles $\mathcal{T}$ with an identification

$$\mathcal{T}^{\otimes r} \xrightarrow{\sim} K\left(-\sum a_i x_i\right).$$

The space of smooth curves endowed with such a line bundle $\mathcal{T}$ is called the *space of r-spin structures* and denoted by $\mathcal{M}^{1/r}_{g;a_1,\dots,a_n}$. It is an unramified r^{2g}-sheeted covering of $\mathcal{M}_{g,n}$.

A compactification of this space, denoted by $\overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n}$, was constructed in [1] and [2] (see also [4] for a slightly simplified version). It is a smooth orbifold (or stack), and there is a finite projection mapping

$$p : \overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n}.$$

The construction uses the so-called *Jarvis-Vistoli twisted curves*, i.e., curves that are themselves endowed with an orbifold structure. The stabilizers of the marked points and the nodes are equal to $\mathbb{Z}/r\mathbb{Z}$, while the stabilizer of any other point is trivial. $\mathcal{T}$ is then an rth root of $K(-\sum a_i x_i)$ *in the orbifold sense.* Alternatively, we can forget about the orbifold structure of the curve and consider only the sheaf of invariant sections of $\mathcal{T}$. We then obtain a rank one torsion-free sheaf rather than a line bundle.

1.2.3 Witten's class.

The rank one torsion-free sheaf (of invariant sections of) $\mathcal{T}$ is defined on the universal curve $\overline{\mathcal{C}}^{1/r}_{g;a_1,\dots,a_n}$ over $\overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n}$. Consider its push-forward to

the space $\overline{\mathcal{M}}_{g;a_1,\ldots,a_n}$ itself. First assume that for each curve C we have $H^0(C,\mathcal{T}) = 0$. Then the spaces $H^1(C,\mathcal{T})$ form a vector bundle $V^\vee$ over $\overline{\mathcal{M}}^{1/r}_{g;a_1,\ldots,a_n}$. We denote by V the dual vector bundle and define Witten's class as

$$c_W(a_1,\ldots,a_n) = c_W = \frac{1}{r^g}\, p_*\, c_{\mathrm{top}}(V).$$

In other words: take the top Chern class of V, push it from $\overline{\mathcal{M}}^{1/r}_{g;a_1,\ldots,a_n}$ to $\overline{\mathcal{M}}_{g,n}$, and divide by r^g. By the Riemann-Roch formula, the (complex) degree of Witten's class is

$$\deg c_W = \frac{(r-2)(g-1) + \sum a_i}{r}.$$

Unfortunately, in general $\mathcal{T}$ has both 0- and 1-cohomologies. The definition of Witten's class c_W in this case is much more involved. There exist two algebro-geometric constructions: see [13] and [3]. (Note that the definition of c_W is special to our situation and uses the identification of $\mathcal{T}^{\otimes r}$ with the canonical bundle. No general constructions from algebraic geometry are expected to work.)

Witten's class satisfies the following vanishing property, that we will use as an axiom:

If one of the a_i's equals $r-1$ then $c_W = 0$.

1.2.4 Factorization rules.

We are interested in the restriction of Witten's class to the boundary components of the moduli space $\overline{\mathcal{M}}_{g,n}$. There are two types of boundary components (see Figure 1): those isomorphic to $\overline{\mathcal{M}}_{g',n'+1} \times \overline{\mathcal{M}}_{g'',n''+1}$, $n'+n''=n$, $g'+g''=g$ and the unique component isomorphic to $\overline{\mathcal{M}}_{g-1,n+2}/\mathbb{Z}_2$.

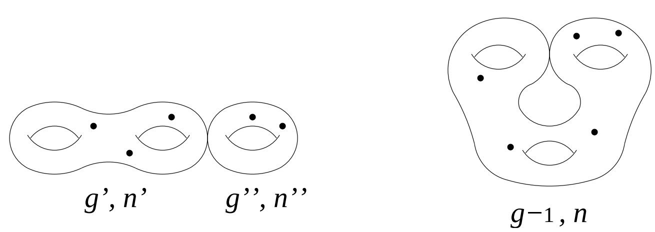

Figure 1: Two possible degenerations of a stable curve.

In the first case, assume for simplicity that the the marked points $x_1,\ldots,x_{n'}$ are on the first component of the curve, while $x_{n'+1},\ldots,x_n$ are on the second

component. There is a unique choice of $a', a'' \in \{0, \dots, r-1\}$ such that

$$2g' - 2 - a' - \sum_{i=1}^{n'} a_i \qquad \text{and} \qquad 2g'' - 2 - a'' - \sum_{i=n'+1}^{n} a_i$$

are both divisible by r. We have $a' + a'' = r - 2$ or $a' = a'' = r - 1$. For the second type of boundary component, we have to sum over all choices of a', a'' such that $a' + a'' = r - 2$. Now we can formulate the factorization rules.

The restriction of Witten's class to the first type boundary component equals

$$c_W(a_1, \dots, a_n) = c_W(a_1, \dots, a_{n'}, a') \times c_W(a_{n'+1}, \dots, a_n, a'').$$

The restriction of Witten's class to the second type boundary component equals

$$c_W(a_1, \dots, a_n) = \frac{1}{2} \sum_{a'+a''=r-2} c_W(a_1, \dots, a_n, a', a'').$$

The vanishing property and the factorization rules (for the Polishchuk-Vaintrob construction) are proved in [14].

1.2.5 Intersection numbers.

We use the standard notation for the intersection numbers of Witten's class with powers of the ψ-classes:

$$\langle \tau_{d_1,a_1} \dots \tau_{d_n,a_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1, \dots, a_n)\, \psi_1^{d_1} \dots \psi_n^{d_n}.$$

Although the genus g is determined by the elements of the bracket by

$$(r+1)(2g-2+n) = \sum (rd_i + a_i + 1),$$

we will sometimes recall it in a subscript. The integer $r \geq 2$ is supposed to be fixed once and for all throughout the paper.

Since double ramification cycles on genus 1 moduli spaces will play a special role, let us introduce them here (this is a particular case of Definition 2.4). Choose n integers $k_1, \dots, k_n$ satisfying $\sum k_i = 0$. We assume that at least one of the k_i's is different from 0.

We will usually assume that the list $(k_1, \dots, k_n)$ starts with the positive integers and ends with the negative ones, the zeroes being in the middle. We will sometimes use the notation

$$(k_1, \dots, k_{n_+} \mid 0, \dots, 0 \mid \tilde{k}_1, \dots, \tilde{k}_{n_-})$$

with only nonnegative integers instead of

$$(k_1, \dots, k_n) = (k_1, \dots, k_{n_+}, 0, \dots, 0, -\tilde{k}_1, \dots, -\tilde{k}_{n_-}).$$

To this list of integers assign the set $D(k_1, \dots, k_n)$ of smooth genus 1 curves $(C, x_1, \dots, x_n)$ such that $\sum k_i x_i$ is the divisor of a function. Let the cycle $\overline{D}(k_1, \dots, k_n)$ be the closure of $D(k_1, \dots, k_n)$.

Definition 1.1 We call $\overline{D}(k_1, \dots, k_n)$ a *double ramification cycle.* The integral of Witten's class over this cycle is denoted by

$$\int\limits_{\overline{D}_{k_1,\dots,k_n}} c_W(a_1, \dots, a_n) = \left\langle \begin{array}{ccc} k_1 & \dots & k_n \\ a_1 & \dots & a_n \end{array} \right\rangle =$$

$$= \left\langle \begin{array}{ccc|ccc|ccc} k_1 & \dots & k_{n_+} & 0 & \dots & 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle$$

Theorem 2 *We have*

$$\left\langle \begin{array}{ccc} k_1 & \dots & k_n \\ a_1 & \dots & a_n \end{array} \right\rangle = \left(\frac{1}{2} \sum_{i=1}^{n} k_i^2 - 1 \right) \cdot \frac{1}{24} \frac{(n-1)!}{r^{n-1}} \prod_{i=1}^{n} (r - 1 - a_i).$$

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2 Preliminaries

2.1 Admissible coverings

Consider a map φ from a smooth curve C to the sphere $S = \mathbb{C}\mathrm{P}^1$. Mark all ramification points on S and all their preimages on C. Now choose several disjoint simple loops on S, that do not pass through the marked points.

Suppose that if we contract these loops we obtain a stable genus 0 curve S'. Now contract also all the preimages of the loops in C to obtain a nodal curve C' that turns out to be automatically stable. We have obtained a map φ' from a nodal curve of genus g to a stable curve of genus 0. It has the same degree over every component of S'. Moreover, at each node of C', the projection φ' has the same local multiplicity on both components meeting at the node.

Definition 2.1 A map from a stable curve of genus g to a stable curve of genus 0 topologically equivalent to a map described above is called an *admissible covering.*

The space of all admissible coverings with prescribed ramification types over the marked points is very useful for the study of moduli spaces (see [6]). It is not normal but can be normalized, and thus one can study its intersection theory. We refer to Ionel's work [6] for detailed definitions. This construction can be slightly extended by allowing additional marked points on the curve C. Their images on the genus 0 curve will also be marked (although they are not ramification points). All the other preimages on C of these new marked points on S should also be marked.

We will be particularly interested in the space of admissible covering with multiple ramifications only over 2 points labeled with 0 and ∞, the other ramification points being simple.

Definition 2.2 Consider the space of admissible coverings of some given genus g with prescribed ramification types $(k_1, \ldots, k_{n_+})$ and $(\tilde{k}_1, \ldots, \tilde{k}_{n_-})$, $\sum k_i = \sum \tilde{k}_i$, over two points labeled 0 and ∞, and with simple ramifications elsewhere. We also suppose that there are n_0 additional marked points on C. The normalization of this space is called a *double ramification space* or a *DR-space.* It is denoted by

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}(k_1, \ldots, k_{n_+}, \overbrace{0, \ldots, 0}^{n_0}, -\tilde{k}_1, \ldots, -\tilde{k}_{n_-}).$$

We do not include the genus g in our notation, although, of course, the space of coverings does depend on g.

If N is the total number of marked points on the curve C, we can consider the forgetful map $j : \bar{\mathbf{A}} \to \overline{\mathcal{M}}_{g,N}$ that forgets the admissible covering retaining only its source curve. Since the curve C is automatically stable, the map j is actually an injection and an isomorphism with its image. The pull-backs by j of the ψ-classes on $\overline{\mathcal{M}}_{g,N}$ coinside with the ψ-classes naturally defined on $\bar{\mathbf{A}}$.

Another forgetful map $f : \bar{\mathbf{A}} \to \overline{\mathcal{M}}_{0,n+2g-2}$, $n = n_+ + n_- + n_0$ takes an admissible to its image genus 0 curve. This map satisfies the following crucial property.

Lemma 2.3 (Ionel's lemma, see [6]) *The map f sends the fundamental homology class of $\bar{\mathbf{A}}$ to (a multiple of) the fundamental homology class of $\overline{\mathcal{M}}_{0,n+2g-2}$. We have*

$$\psi_i(\bar{\mathbf{A}}) = \frac{1}{k_i}\psi_0(\overline{\mathcal{M}}_{0,n+2g-2}), \qquad \psi_{n_++n_0+i}(\bar{\mathbf{A}}) = \frac{1}{\tilde{k}_i}\psi_\infty(\overline{\mathcal{M}}_{0,n+2g-2}).$$

In other words, a ψ-class on $\bar{\mathbf{A}}$ at a marked zero or pole of the admissible covering coincides, up to a constant, with the pull-back of the ψ-class at 0 *or* ∞ *of the genus 0 moduli space.*

Finally, let us define double ramification cycles. For a given moduli space $\overline{\mathcal{M}}_{g,n}$ choose an integer p, $0 \le p \le g$ and $n+p$ integers k_i. We suppose that $\sum k_i = 0$ and that none of the $k_{n+1}, \dots, k_{n+p}$ vanishes.

Definition 2.4 Consider the set of smooth curves $(C, x_1, \dots, x_n) \in \mathcal{M}_{g,n}$ such that there exist p more marked points $x_{n+1}, \dots, x_{n+p}$ and a meromorphic function on C with no zeroes or poles outside of $x_1, \dots, x_{n+p}$, the orders of zeroes or poles being prescribed by the list $k_1, \dots, k_{n+p}$ ($k_i > 0$ for the zeroes, $k_i < 0$ for the poles, and $k_i = 0$ for the marked points that are neither zeroes nor poles). The closure of this set in $\overline{\mathcal{M}}_{g,n}$ is called the *double ramification cycle* or a *DR-cycle*.

By a generalization of Mumford's argument in [11], one can show that the codimension of a DR-cycle is equal to $g - p$ whenever there is at least one positive and one negative number among $k_1, \dots, k_n$. Assuming that this condition is satisfied we see that for $p = g$ the DR-cycle coincides with the moduli space $\overline{\mathcal{M}}_{g,n}$. Faber and Pandharipande [5] proved that the cohomology classes Poincaré dual to any DR-cycle belongs to the tautological ring of the moduli space of curves. Their proof may be used to obtain new relations for intersection numbers with Witten's class.

It follows that the forgetful map $h : \bar{\mathbf{A}}(k_1, \dots, k_{n+p}) \to \overline{\mathcal{M}}_{g,n}$ sends the fundamental comology class of $\bar{\mathbf{A}}(k_1, \dots, k_{n+p})$ to (a multiple of) the fundamental homology class of the DR-cycle.

Conventions. Figure 2 represents a DR-space

$$\bar{\mathbf{A}}(k_1 + 1, \dots, k_{n_+}, 0, \dots, 0, -\tilde{k}_1, \dots, -\tilde{k}_{n_-}, -1).$$

It also shows the corresponding maps j, f and h. In this figure, as well as in the subsequent figures and in the text we follow the following conventions. A **cross** represents a critical point or a ramification point of an admissible covering. **Round black dots** represent the marked points $x_1, \ldots, x_n$ (they are not forgotten under the map h). The images of these points under the maps j, f, and h are also represented as round black dots. **Square black dots** represent the points $x_{n+1}, \ldots, x_{n+p}$ (they are forgotten under the map h). Finally, **white dots** represent all the marked points on the curve C different from the critical points and from $x_1, \ldots, x_{n+p}$. We will not show them in the figures when it is not necessary.

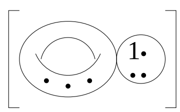
2.2 Intersection numbers in genus 0

In our computations we will need the value of the bracket

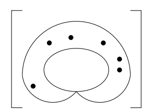
$$\langle \tau_{1,a_1} \tau_{0,a_2} \ldots \tau_{0,a_n} \rangle = \int_{\overline{\mathcal{M}}^{1/r}_{g;a_1,\ldots,a_n}} c_W \psi_1$$

for $\sum a_i = (n-1)r$. The *topological recursion relation* expresses ψ_1 as a sum of divisors:

$$\psi_1 =$$



$$+ \frac{1}{12}$$



$. \qquad (1)$

We must integrate Witten's class over each of these divisors. The first divisor contributes 0 [1] while the second one contributes

$$\frac{1}{24} \sum_{a'+a''=r-2} \langle \tau_{0,a_1} \ldots \tau_{0,a_n} \tau_{0,a'} \tau_{0,a''} \rangle_0 \, .$$

This is a combination of integrals of Witten's class (without ψ-classes) over genus 0 moduli spaces, and we are now going to determine its value. For simplicity, in this section we will use nonstandard notation for the bracket: $\langle a_1, \ldots, a_n \rangle$ instead of $\langle \tau_{0,a_1} \ldots \tau_{0,a_n} \rangle$.

Proposition 2.5 *For any $m \le r-2$ and for any $x_1, \ldots, x_n$, $0 \le x_i \le r-2$, $\sum x_i = nr - m - 2$, we have*

$$\sum_{a+b=m} \langle a, b, x_1, \ldots, x_n \rangle = \frac{(n-1)!}{r^{n-1}} \prod_{i=1}^{n} (r-1-x_i).$$

[1] Indeed, the integral of c_W over this divisor includes as a factor the integral of c_W over a genus 1 moduli space, which vanishes for reasons of dimension.

Proof. The proof uses the known initial values of the bracket:

$$\begin{array}{rcll} \langle a_1, a_2, a_3 \rangle & = & 1 & \text{for } \sum a_i = r-2, \\ \langle a_1, a_2, a_3, a_4 \rangle & = & \frac{1}{r}\min(a_i, r-1-a_i) & \text{for } \sum a_i = 2r-2, \end{array}$$

and the WDVV equation. First note that the proposition is true for $n = 1$ (it says that $\sum_{a+b=m} 1 = m+1$).

Now we assume $n \geq 2$ and proceed by induction on m.

If $m = 0$ then the only possible bracket is $\langle 0, 0, r-2 \rangle$.

Suppose that the proposition is true up to some m. Let us apply the WDVV equation to the correlators containing $1, a, b, x_1, \dots, x_n$, the four distinguished points being 1, a, b, and x_1. In other words, we consider all possible degenerations of the sphere into two components such that 1 and a lie on one component, while b and x_1 lie on the other one, and then we swap the four points. We obtain the following equality. (The summation over $a + b = m$ is implicitly assumed; a hat means that the symbol is skipped; underlined symbols are not in the original list, they appear at the node of the degenerate sphere in the WDVV formula.)

$$\langle 1, a, \underline{r-3-a} \rangle \, \langle \underline{a+1}, b, x_1, \dots, x_n \rangle \tag{A}$$

$$+ \sum_{i \neq 1} \Big\langle 1, a, x_i, \underline{2r-3-a-x_i} \Big\rangle \, \Big\langle \underline{a+1+x_i-r}, b, x_1, \dots, \widehat{x_i}, \dots, x_n \Big\rangle \tag{B}$$

$$= \Big\langle 1, x_1, \underline{r-3-x_1} \Big\rangle \, \Big\langle a, b, \underline{x_1+1}, x_2, \dots, x_n \Big\rangle \tag{C}$$

$$+ \sum_{i \neq 1} \Big\langle 1, x_1, x_i, \underline{2r-3-x_1-x_i} \Big\rangle \times$$

$$\times \Big\langle a, b, \underline{x_1+x_i+1-r}, x_2, \dots, \widehat{x_i}, \dots, x_n \Big\rangle . \tag{D}$$

The term (A) is the sum over $(a+1)+b = m+1$ that we want to determine. The missing case $a+1 = 0$ does not matter, because for $n \geq 2$ a single zero entry makes a bracket vanish.

The term (B) can be evaluated by the induction assumption. It vanishes if $a+1+x_i < r$, while for $a+1+x_i \geq r$ we obtain a sum over $(a+1+x_i)+b = m + x_i - r < m$. Thus we have

$$(B) = \frac{1}{r} \cdot \frac{(n-2)!}{r^{n-2}} \prod_{j \neq i} (r-1-x_j).$$

The term (C) can, once again, be evaluated by the induction assumption:

$$(C) = \frac{(n-1)!}{r^{n-1}} (r-2-x_1) \prod_{j \neq 1} (r-1-x_j).$$

The last term (D) vanishes if $x_1 + x_i + 1 < r$. Luckily, it turns out that for any $i > 1$, we have $x_1 + x_i + 1 \geq r$. Indeed, if $a + b = m \leq r - 3$ and $x_1 + x_j \leq r - 2$, then the sum of the $n - 2$ remaining terms equals $nr - m - 2 - x_1 - x_i \geq (n-2)r + 3$, which is impossible since each of them equals at most $r - 2$. Therefore we have

$$(D) = \frac{1}{r} \cdot \frac{(n-2)!}{r^{n-2}} [(r - 1 - x_1) + (r - 1 - x_i)] \prod_{j \neq 1,i} (r - 1 - x_j).$$

We deduce that

$$(A) = (C) + (D) - (B) = \frac{(n-1)!}{r^{n-1}} \prod_j (r - 1 - x_j).$$

$\diamondsuit$

Remark 2.6 By looking through the proof carefully one can check that the formula given in the proposition actually holds for $m \leq r$, except if $n = 1$.

3 Computations with double ramification cycles

In this section we prove Theorem 2. By algebro-geometric arguments we find several relations binding the values of the brackets involved in the theorem. In Section 3.1 we write down these relations and prove that they suffice to determine the values of the bracket in all cases. We prove the relations in Section 3.2.

3.1 The relations

Denote by

$$B = \langle \tau_{1,a_1} \tau_{0,a_2} \ldots, \tau_{0,a_n} \rangle_{g=1} = \frac{1}{24} \frac{(n-1)!}{r^{n-1}} \prod_{i=1}^{n} (r - 1 - a_i)$$

(see Section 2.2).

Then the following relations hold.

Relation 1.

$$
\begin{aligned}
&(k_1+1)(n_+ + n_- + 1)B \;=\\
&-(k_1+n_++n_-+1)\left\langle \begin{array}{ccc|c|ccc} k_1 & \dots & k_{n_+} & 0\ \dots\ 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle\\
&-\sum_{i=2}^{n_+}(k_i-1)\left\langle \begin{array}{cccccc|c|ccc} k_1+1 & k_2 & \dots & k_i-1 & \dots & k_{n_+} & 0\ \dots\ 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & & & \dots & & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle\\
&+\sum_{i=1}^{n_-}(\tilde{k}_i+1)\left\langle \begin{array}{cccc|c|ccccc} k_1+1 & k_2 & \dots & k_{n_+} & 0\ \dots\ 0 & \tilde{k}_1 & \dots & \tilde{k}_i+1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & & \dots & \dots & \dots & & \dots & & a_n \end{array} \right\rangle .
\end{aligned}
$$

Relation 2.

$$
\begin{aligned}
(k_1+1)B \;=& -\left\langle \begin{array}{ccc|c|ccc} k_1 & \dots & k_{n_+} & \overbrace{0\ \dots\ 0}^{n_0} & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle\\
&+\left\langle \begin{array}{cccc|c|cccc} k_1+1 & k_2 & \dots & k_{n_+} & \overbrace{0\ \dots\ 0}^{n_0-1} & 1 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle .
\end{aligned}
$$

Relation 3.

$$
\left\langle \begin{array}{c|c|c} 1 & 0\ \dots\ 0 & 1 \\ a_1 & \dots & a_n \end{array} \right\rangle = 0.
$$

To these relations we may add the simple observation that the brackets are invariant under renumberings of the marked points and under a simultaneous change of sign of all k_i's.

One can check by direct computations that these relations are compatible with the expression of the bracket given in Theorem 2.

Lemma 3.1 *Relations 1, 2, and 3 determine the values of all brackets unambiguously.*

Proof. The lemma is proved by induction on the number of nonzero entries $n_+ + n_-$ in a bracket, the base of induction being given by Relation 3.

Consider a bracket whose value we would like to determine.

First case: suppose that one of the $\tilde{k}_i$'s is equal to 1 and one of the k_i's is greater than 1 (or vice versa). Then we take our bracket to be the last term in Relation 2 and replace it by the first term (plus a know multiple of B). We have decreased the number of nonzero k_i's in the bracket to be computed.

Second case: suppose that all the (nonzero) k_i's and $\tilde{k}_i$'s equal 1. In particular, $n_+ = n_-$. If $n_+ = n_- = 1$ the bracket vanishes by Relation 3. If $n_+ = n_- \geq 2$, we take our bracket to be the first term in Relation 1. Our bracket is than replaced by a sum of brackets in which one of the k_i's and one of the $\tilde{k}_i$'s are equal to 2, while all the others are equal to 1. Such brackets fall into the first case.

Third case: suppose that all of the k_i's and $\tilde{k}_i$'s are greater than 1. Then we take our bracket to be the first term of the second sum in Relation 1 and replace it by the sum of the other terms. As a result we obtain a combination of brackets in all of which the number $\tilde{k}_1$ is smaller than in the initial bracket. Repeating the same operation for each bracket we are sure that $\tilde{k}_1$ will decrease by 1 with every step. Thus after a finite number of steps we will end up with a collection of brackets all of which fall into the first two cases. $\diamondsuit$

Thus the relations determine the values of the brackets unambiguously. Since they are compatible with the expression of Theorem 2, the theorem will be completely proved once we will have proved the relations.

3.2 Proof of the relations

First of all, note that Relation 3 is obvious. The bracket in this relation denotes an integral over an empty space, therefore it vanishes. Thus we only need to prove Relations 1 and 2.

Consider the commutative diagram involving four moduli spaces shown in Figure 2.

Here $\bar{\mathbf{A}} = \bar{\mathbf{A}}(k_1+1, \ldots, k_n, -1)$ is the DR-space of genus 1 admissible coverings with ramification types $(k_1+1, k_2, \ldots, k_{n_+})$ and $(1, \tilde{k}_1, \ldots, \tilde{k}_{n_-})$ over 0 and ∞. There are also n_0 additional marked points on the source curve.

The projection π forgets all marked points except the round black dots and stabilizes the curve.

In this example, the DR-cycle coincides with the whole moduli space $\overline{\mathcal{M}}_{1,n}$.

The main technique of this section is to compute the integral

$$I = \int_{\bar{\mathbf{A}}} h^*(c_W)\psi_1$$

by three different methods and to use the relations thus obtained. The three methods can be summarized as follows.

$$\begin{tikzcd}
\bar{\mathbf{A}} \arrow[r, "j"] \arrow[d, "f"'] \arrow[dr, "h"] & \overline{\mathcal{M}}_{1,(n+1)(K+1)} \arrow[d, "\pi"] \\
\overline{\mathcal{M}}_{0,n+3} & \overline{\mathcal{M}}_{1,n}
\end{tikzcd}$$

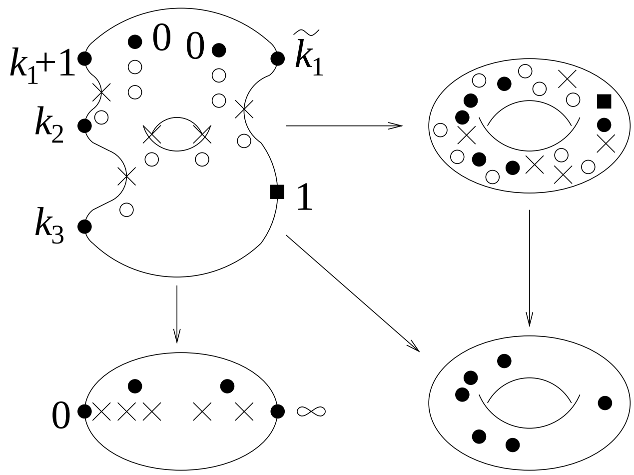


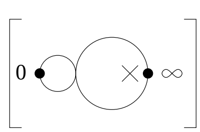
Figure 2: Four moduli spaces.

A. Express the ψ-class on $\overline{\mathcal{M}}_{0,n+3}$ as a sum of boundary divisors. By Ionel's lemma (Lemma 2.3), the preimages under f of these boundary divisors represent the class ψ_1 on $\bar{\mathbf{A}}$. The images of these divisors under h turn out to to be DR-cycles. Therefore the integrals of Witten's class over these divisors are some values of the bracket of Definition 1.1.
B. Same as (A) with another expression of the ψ-class in terms of boundary divisors.
C. In Section 2.2 we computed the integral

$$B = \int_{\overline{\mathcal{M}}_{1,n}} c_W \psi_1$$

analogous to I. Now, the classes ψ_1 on $\overline{\mathcal{M}}_{1,n}$ and on $\bar{\mathbf{A}}$ differ by a sum of boundary divisors. The images of these divisors under h turn out, once again, to be DR-cycles, and integrating Witten's class upon them we obtain a linear combination of brackets.

A. On the space $\overline{\mathcal{M}}_{0,n+3}$ in Figure 2, consider the class ψ at the point labeled by 0. This class is equal to a sum of boundary divisors that we will represent as

$\psi =$ 

. (2)

The picture represents a sum of boundary divisors where the sphere splits into two components. The point labeled by 0 is on the first component, while

the point labeled by ∞ together with some *chosen* critical value (say, the first one) is on the second component. The other marked points on the sphere can be distributed arbitrarily between the components.

Consider the preimages under f of these boundary strata.

Lemma 3.2 *Among the preimages in* $\bar{\mathbf{A}}$ *of the divisors* (2) *consider those on which the integral of* $h^*(c_W)$ *does not vanish. These divisors are* DIV_1, DIV_i, *and* $\widetilde{\mathrm{DIV}}_i$ *in Figure 3.*

It should be understood that each picture actually represents a *generic* admissible covering lying in the divisor. In other words, the divisor is the closure of the set of admissible coverings with the topological structure shown in the figure.

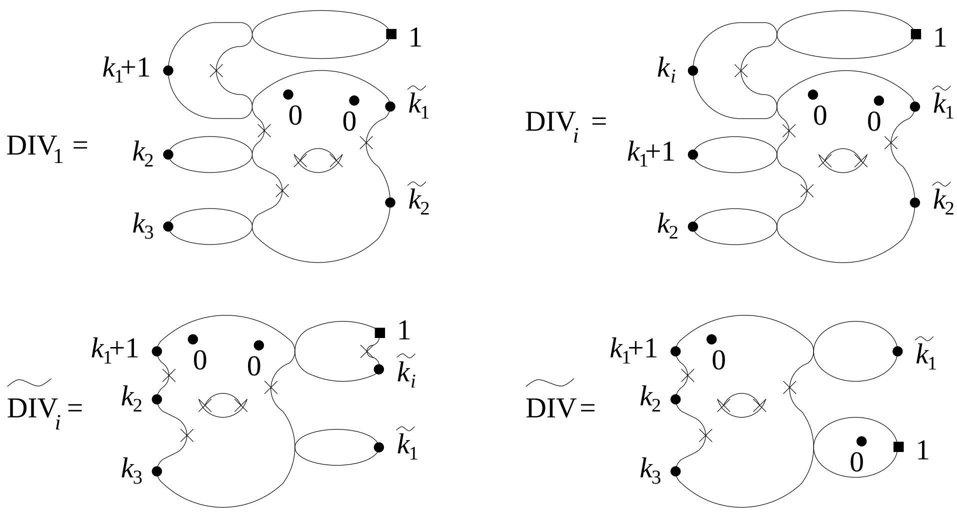
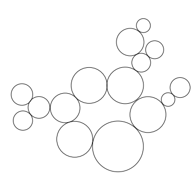


Figure 3: Divisors in $\bar{\mathbf{A}}$.

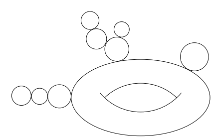
Proof of the lemma. We consider an irreducible component of the preimage of (2) and reason in terms of the generic admissible covering $\varphi : C \to S$ in this component.

First suppose that the curve C does not have a toric component. Then it looks as a ring of spheres with several "tails":

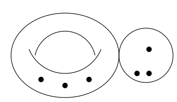


Each sphere of the ring contains at least 2 nodes. Thus the restriction of φ to any such sphere is of degree at least 2. Which implies that there is at least one round black dot on each sphere of the ring (there is at least one preimage of 0 or ∞ on each component, and the square black dot alone, being a simple preimage, is not enough for φ to have degree 2). It follows that none of the spheres of the ring is contracted under the map h to $\overline{\mathcal{M}}_{1,n}$. Now, the number of spheres in the ring is even (those that contain a preimage of 0 alternate with those that contain a preimage of ∞). Thus the curve C retains at least 2 nodes after the projection to $\overline{\mathcal{M}}_{1,n}$. Which means that the codimension of the image of the corresponding divisor under h is at least 2, so the integral of Witten's class on it vanishes. The conclusion is that we only need to consider curves C with a toric component.

Suppose C has a toric component with several "tails":



If one of the tails contains more than one round black dot, than the projection h of such a curve to $\overline{\mathcal{M}}_{1,n}$ looks like:



The integral of Witten's class over the divisor of such curves vanishes. The conclusion is that each "tail" contains at most one round black dot.

Note that every component of C contains at least one black dot (round or square): either a preimage of 0 or of ∞. It follows that each tail (except perhaps one) is composed of exactly 1 sphere containing exactly 1 black dot. The exceptional tail can be composed of 1 or 2 spheres containing a round black dot and a square black dot. On each simple tail the function φ has the form $z \mapsto z^k$ for some positive integer k.

Actually, an exceptional tail is bound to exist. Indeed, suppose there is no exceptional tail, i.e., every tail is composed of 1 sphere with a unique black dot. The image curve S has 2 components, and in the case we consider, the preimage of one of them is the toric component of C, while the preimage of the other is the union of the tails. Then the latter component of S contains no marked points except 0 or ∞. This is impossible, because S is stable.

If the exceptional tail is composed of 1 sphere, then this sphere contains the black square dot and some dot $\tilde{k}_i$. This gives the divisor $\widetilde{\mathrm{DIV}}_i$. If the exceptional tail is composed of 2 spheres, then one of them contains the

square black dot, and the other one some dot k_i or k_1+1. The sphere with the square dot can contain at most one node because the degree of φ on it equals 1. Thus we obtain the divisors DIV_1 and DIV_i.

The lemma is proved. $\diamondsuit$

Now, the integral I is the sum of integrals of Witten's class over the divisors DIV_1, DIV_i, and $\widetilde{\mathrm{DIV}}_i$, which we will now determine.

Lemma 3.3 *The contributions to I of the divisors DIV_1, DIV_i, and $\widetilde{\mathrm{DIV}}_i$ equal*

$$\frac{(n_+ + n_-)!(K-1)!^{n_++n_-+1}K!^{n_0}}{k_1+1} \cdot \; k_1(n_++n_-) \; \times$$

$$\times \left\langle \begin{array}{ccc|ccc|ccc} k_1 & \dots & k_{n_+} & 0 & \dots & 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & & \dots & & \dots & \dots & a_n \end{array} \right\rangle,$$

$$\frac{(n_+ + n_-)!(K-1)!^{n_++n_-+1}K!^{n_0}}{k_1+1} \cdot \; (k_i-1)(n_++n_-) \; \times$$

$$\times \left\langle \begin{array}{cccccc|ccc|ccc} k_1+1 & k_2 & \dots & k_i-1 & \dots & k_{n_+} & 0 & \dots & 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & & & \dots & & \dots & & \dots & & \dots & \dots & a_n \end{array} \right\rangle,$$

$$\frac{(n_+ + n_-)!(K-1)!^{n_++n_-+1}K!^{n_0}}{k_1+1} \cdot \; (\tilde{k}_i+1) \; \times$$

$$\times \left\langle \begin{array}{cccc|ccc|ccccc} k_1+1 & k_2 & \dots & k_{n_+} & 0 & \dots & 0 & \tilde{k}_1 & \dots & \tilde{k}_i+1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & & \dots & & \dots & & \dots & & \dots & & a_n \end{array} \right\rangle,$$

respectively, where $K = \sum k_i = \sum \tilde{k}_i$.

Proof. The fundamental classes of the divisors DIV_1, DIV_i, and $\widetilde{\mathrm{DIV}}_i$ project to multiples of double ramification divisors in $\overline{\mathcal{M}}_{1,n}$ under the map h. Thus we can integrate Witten's class over these double ramification divisors, which explains the brackets that appear in the answers.

The coefficients in front of these brackets arise as products of three factors: (i) the transversal multiplicity of the map f on the divisor (because we need pull-backs under f of homology classes rather than geometric pull-backs); (ii) the degree of h on the divisor; (iii) the factor $1/(k_1+1)$ coming from Ionel's lemma. For example, for DIV_i, these factors are:

$$\text{(i)} \quad (k_1+1)k_2 \dots k_{i-1}(k_i-1)k_{i+1}\dots k_{n_+}.$$

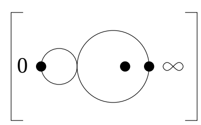
These factors come from the number of ways to resolve the nodes of the source curve.

$$\text{(ii)} \quad \frac{(n_+ + n_-)(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{(k_1+1)k_2 \dots k_{i-1}k_{i+1} \dots k_{n_+}}.$$

The factor $(n_+ + n_-)(n_+ + n_-)!$ is the number of ways to number the ramification points on the two-component genus 0 image curve, taking into account that the first ramification point lies on the component of ∞. $(K-1)!^{n_+ + n_- + 1}$ is the number of ways to number the white (noncritical) preimages of the ramification points. $K!^{n_0}$ is the number of ways to number the white preimages of the black marked points different from 0 and ∞. The denominator comes from the spheres on the left of the picture of DIV_i: the restriction of the admissible covering map φ on these spheres has the form z^k, and thus $z \mapsto (-1)^{1/k} z$ gives a renumbering of the white dots on such a sphere equivalent to the initial numbering.

Multiplying these factors (without forgetting $1/(k_1+1)$) we obtain the coefficient for DIV_i as claimed in the lemma. The computations for the other divisors are analogous. $\diamondsuit$

B. The second way of computing the integral I is not very different from the first one. This time we start with a different presentation of the ψ-class at the point labeled by 0 on $\overline{\mathcal{M}}_{0,n+3}$:

$\psi =$  . (3)

The picture represents a sum of boundary divisors where the sphere splits into two components. The point labeled by 0 is on the first component, while the point labeled by ∞ together with another *chosen* black marked point (say, the first one) is on the second component.

Lemma 3.4 *Among the preimages in* $\bar{\mathbf{A}}$ *of the divisors* (3) *consider those on which the integral of* c_W *does not vanish. These divisors are* DIV_1, DIV_i, *and* $\widetilde{\mathrm{DIV}}$ *in Figure 3.*

Proof. First suppose there are no points marked with crosses on the component of ∞ on the image curve. This means that the preimages of this component are spheres containing only black marked points. However, all these components must be contracted under the map h. (Otherwise the integral of c_W over this divisor vanishes.) Therefore, actually, every sphere

contains exactly one black point, except one sphere that contains a round black dot and a square black dot. This is the divisor $\widetilde{\mathrm{DIV}}$.

If the component of ∞ in the image curve contains at least one cross-marked point, then we have reduced the problem to the situation of Lemma 3.2. Thus the only divisors that can give a nonzero contribution are DIV_1, DIV_i, and $\widetilde{\mathrm{DIV}}_i$. However, actually the divisors $\widetilde{\mathrm{DIV}}_i$ do not appear as preimages of the divisors (3) because in this case the component of ∞ in the image contains no black dots other than ∞. $\diamondsuit$

Lemma 3.5 *The contributions to I of the divisors DIV_1, DIV_i, and $\widetilde{\mathrm{DIV}}$ equal*

$$\frac{(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{k_1+1} \cdot k_1(n_+ + n_- + 1) \times$$

$$\times \left\langle \begin{array}{ccc|ccc|ccc} k_1 & \dots & k_{n_+} & 0 & \dots & 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & & \dots & & \dots & \dots & a_n \end{array} \right\rangle,$$

$$\frac{(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{k_1+1} \cdot (k_i - 1)(n_+ + n_- + 1) \times$$

$$\times \left\langle \begin{array}{cccccc|ccc|ccc} k_1+1 & k_2 & \dots & k_i - 1 & \dots & k_{n_+} & 0 & \dots & 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & & & \dots & & \dots & & \dots & & \dots & \dots & a_n \end{array} \right\rangle,$$

$$\frac{(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{k_1+1} \cdot (n_+ + n_- + 1) \times$$

$$\times \left\langle \begin{array}{cccc|c|cccc} k_1+1 & k_2 & \dots & k_{n_+} & \overbrace{0 \dots 0}^{n_0 - 1} & 1 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle,$$

respectively, where $K = \sum k_i = \sum \tilde{k}_i$.

The proof is analogous to that of Lemma 3.3. $\diamondsuit$

C. Now we are going to evaluate

$$I = \int_{\bar{\mathbf{A}}} h^*(c_W)\psi_1(\bar{\mathbf{A}})$$

using the value

$$B = \int_{\overline{\mathcal{M}}_{1,n}} c_W \psi_1(\overline{\mathcal{M}}_{1,n}).$$

If the class ψ_1 on $\bar{\mathbf{A}}$ were a pull-back of the class ψ_1 on $\overline{\mathcal{M}}_{1,n}$, then we would simply have $I = B \cdot \deg h$. However this equality is actually incorrect, because there is difference between $h^*(\psi_1(\overline{\mathcal{M}}_{1,n}))$ and $\psi_1(\bar{\mathbf{A}})$.

Lemma 3.6 *The difference $\psi_1(\bar{\mathbf{A}}) - h^*(\psi_1(\overline{\mathcal{M}}_{1,n})$ can be represented as the sum of divisors*

$$k_2 \dots k_{n_+} \, \mathrm{DIV}_1 \; + \; \sum_{i=2}^{n_+} k_2 \dots k_{i-1}(k_i - 1)k_{i+1} \dots k_{n_+} \, \mathrm{DIV}_i$$

(*the divisors being shown in Figure* 3) *and, in addition, some other divisors on which the integral of c_W vanishes.*

Proof. There is a subtlety in finding the difference $\psi_1(\bar{\mathbf{A}}) - h^*(\psi_1(\overline{\mathcal{M}}_{1,n})$. First consider the difference $\psi_1(\overline{\mathcal{M}}_{1,(n+1)(K+1)}) - \pi^*(\psi_1(\overline{\mathcal{M}}_{1,n})$ in the upper right moduli space in Figure 2. It is given by the divisor D_1 of all stable curves on which the first black marked point $(k_1 + 1)$ is situated on a component contracted by π. Now we consider the intersection of this divisor with the image $j(\bar{\mathbf{A}})$. It turns out that this intersection is not necessarily transversal, and its multiplicity gives the coefficients that appear in the formulation of the lemma.

Now we take the pull-back of the intersection by j. Let us consider an irreducible component of this pull-back

$$j^* \Big(\psi_1(\overline{\mathcal{M}}_{1,(n+1)(K+1)}) - \pi^*(\psi_1(\overline{\mathcal{M}}_{1,n})) \Big)$$

and a generic admissible covering φ in this component.

The image curve of φ in $\overline{\mathcal{M}}_{0,n+3}$ has a unique node (otherwise the codimension of such a component in $\bar{\mathbf{A}}$ would be at least 2).

Moreover, this node separates 0 and ∞. Indeed, otherwise the component containing the point $k_1 + 1$ also contains some preimage of ∞. Since this component is contracted by h, the only possible preimage is the square black dot (and there are no other preimages of ∞). Thus the degree of φ on this component equals 1. But this implies $k_1 + 1 = 1$, which is impossible.

On the component of ∞ there exists at least one cross marked point. Indeed, otherwise there is no ramification over this component, so all the ramification is over the component containing 0. Then the point $k_1 + 1$ lies on the torical component, which is impossible, since this component should be contracted by h.

Thus we have reduced the problem to the situation of Lemma 3.2 with the additional restriction that the component containing the point $k_1 + 1$

should be contracted by h. This restriction excludes the divisors $\widetilde{\mathrm{DIV}}_i$ but allows the divisors DIV_1 and DIV_i.

It remains to find the multiplicity of the intersection $j(\bar{\mathbf{A}}) \cap D_1$ along $j(\mathrm{DIV}_1)$ and $j(\mathrm{DIV}_i)$. This multiplicity is easily seen to be equal to the product of indices of the nodes that are desingularized as a generic point of the intersection moves to a generic point of D_1. Such products of indices are precisely the coefficients stated in the lemma. $\diamondsuit$

Lemma 3.7 *The contributions to I of the integral B and that of the divisors DIV_1 and DIV_i equal*

$$\frac{(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{k_1+1} \quad \cdot \quad (k_1+1)(n_+ + n_- + 1) \quad \cdot \quad B,$$

$$\frac{(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{k_1+1} \quad \cdot \quad k_1(n_+ + n_- + 1) \ \times$$

$$\times \left\langle \begin{array}{ccc|c|ccc} k_1 & \dots & k_{n_+} & 0 \dots 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & \dots & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle,$$

$$\frac{(n_+ + n_-)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{k_1+1} \quad \cdot \quad (k_i-1)(n_+ + n_- + 1) \ \times$$

$$\times \left\langle \begin{array}{cccccc|c|ccc} k_1+1 & k_2 & \dots & k_i - 1 & \dots & k_{n_+} & 0 \dots 0 & \tilde{k}_1 & \dots & \tilde{k}_{n_-} \\ a_1 & & & \dots & & \dots & \dots & \dots & \dots & a_n \end{array} \right\rangle,$$

respectively, where $K = \sum k_i = \sum \tilde{k}_i$.

Proof. The coefficient of B is just the degree of h. The coefficients of the other two brackets are obtained as products of two factors: (i) the coefficients of the divisors appearing in Lemma 3.6, (ii) the degree of h on the divisor.

For instance, for DIV_i these factors equal:

$$\text{(i)} \quad k_2 \dots k_{i-1}(k_i - 1)k_{i+1} \dots k_{n_+}.$$

$$\text{(ii)} \quad \frac{(n_+ + n_- + 1)!(K-1)!^{n_+ + n_- + 1} K!^{n_0}}{(k_1+1)k_2 \dots k_{i-1}k_{i+1} \dots k_{n_+}}.$$

The case of the divisor DIV_1 is analogous. $\diamondsuit$

Thus we have established three expressions, A, B, and C for the integral I. Writing $\mathrm{A} - \mathrm{B} = 0$ and $\mathrm{A} - \mathrm{C} = 0$ we obtain Relations 1 and 2.

4 An algorithm to compute Witten's intersection numbers

Here we present an algorithm for computing any number $\langle \tau_{d_1,a_1} \dots \tau_{d_n,a_n} \rangle$. This algorithm is rather hard to implement on a computer because it involves enumerating all possible degenerations of an admissible covering satisfying some given properties. Thus this section is best viewed as a constructive proof of Theorem 1.

Suppose we are given n nonnegative integers $d_1, \dots, d_n$. Choose an integer p, $0 \le p \le g$ and n numbers $k_1, \dots, k_n$ large enough (we will use the explicit bound $|k_i| > \sum d_i$) such that $\sum k_i = p$. Consider the DR-space $\bar{\mathbf{A}} = \bar{\mathbf{A}}(k_1, \dots, k_n, -1, \dots, -1)$, where the list ends with p numbers -1 so that the total sum is 0 as it should be.

As explained in Section 2.1, there is a projection $h : \bar{\mathbf{A}} \to \overline{\mathcal{M}}_{g,n}$ that sends $\bar{\mathbf{A}}$ to DR-cycle $\overline{D}$ of codimension p in $\overline{\mathcal{M}}_{g,n}$.

We are going to compute the integral

$$\int_{\bar{\mathbf{A}}} h^*(c_W\, \psi_1^{d_1} \dots \psi_n^{d_n}) = \deg(h) \int_{\overline{D}} c_W\, \psi_1^{d_1} \dots \psi_n^{d_n}. \tag{4}$$

If $p = g$ this will give us the value of the bracket $\langle \tau_{d_1,a_1} \dots \tau_{d_n,a_n} \rangle$. Note, however that our result is actually more general.

Our algorithm for computing Integral (4) can be summed up as follows.

While the class ψ_i on $\overline{\mathcal{M}}_{g,n}$ is, in general, not representable by boundary divisors, its pull-back to $\bar{\mathbf{A}}$ can be easily represented as a sum like that. At each step we will consider one ψ-class and replace it by a sum of divisors. Thus we will be reduced to computing the integral of a product $c_W \cdot$ (powers of ψ-classes) over some divisors, the number of ψ-classes having decreased by 1. Using the factorization rules for Witten's class, we will be able to represent each integral over a boundary divisor as a product of analogous integrals over simpler DR-spaces.

Proposition 4.1 *The integral* (4) *can be effectively expressed as a combination of analogous integrals with a smaller sum* $\sum d_i$.

Proof.
1. Expressing the pull-back $h^*(\psi_i)$ as a sum of boundary divisors of $\bar{\mathbf{A}}$.

Consider the class $\psi_i = \psi_i(\overline{\mathcal{M}}_{g,n})$ on $\overline{\mathcal{M}}_{g,n}$ and the corresponding class $\Psi_i = \psi_i(\bar{\mathbf{A}})$ on $\bar{\mathbf{A}}$. The pull-back $h^*(\psi_i)$ to $\bar{\mathbf{A}}$ can be represented as a sum of divisors in the following way. First, the class Ψ_i can, by Ionel's lemma (Lemma 2.3), be replaced by the pull-back of the ψ-class at 0 or at ∞ on

$\overline{\mathcal{M}}_{0,n+p+2g}$. The latter class is easy to represent by a sum of divisors, see Equation (2). Second, the difference $\Psi_i - h^*(\psi_i)$ is equal to the sum (with certain coefficients) of divisors D_i formed by the admissible coverings for which the component of the source curve containing the ith marked point is contracted by h (cf. Lemma 3.6).

2. The image curve of a generic admissible covering in every boundary divisor has a unique node separating 0 and ∞.

Let us consider an irreducible component of one of the above divisors and a generic admissible covering φ in this component. The image curve of φ in $\overline{\mathcal{M}}_{0,n+p+2g}$ has a unique node because otherwise the codimension of such a component in $\bar{\mathbf{A}}$ would be at least 2.

Let us prove that this node separates 0 and ∞. Indeed, this is obvious by construction for the divisors involved in the expression of Ψ_i. As for the divisors representing the difference $\Psi_i - h^*(\psi_i)$, suppose that the node on the image curve does not separate 0 and ∞. Consider the component of the source curve C containing the ith black round marked point. This compounent must contain both a preimage of 0 and a preimage of ∞. On the other hand, it is, by construction, contracted by h. Thus the only possibility is that it contains a unique round black dot (a preimage of 0) and one or several square black dots (since these are forgotten by h). But then the degree of φ on this component is equal to $|k_i|$ and, at the same time, to the number of square black dots. Since we assumed that $|k_i| > \sum d_i$ this is impossible.

3. Splitting the admissible covering.

Denote by DIV the boundary divisor of $\bar{\mathbf{A}}$ under consideration.

Since a generic admissible covering in DIV can have nontrivial ramifications only over 0, ∞, and the node of the image curve, we see that there are at most 2 multiple ramification points on each component of the image curve. Thus, we can split the admissible covering according to the components of the source curve and obtain several simpler admissible coverings lying in simpler DR-spaces. This is shown in Figure 4 (the meaning of various markings will be explained later). We will call the components of the source curve in a generic admissible covering *parts*. In Figure 4 the parts are denoted by A, B, C, D, and E.

The smaller admissible coverings will have a lot of unnecessary white marked points. Consider, for instance, a critical point (a cross) on the part D in Figure 4. Its image is a ramification point (a cross) in the image curve. This cross has several simple (white) preimages on the parts B and E. Once we have separated the parts we would like to forget these useless white points. This is done in the following way: (i) forget those marked points on the image

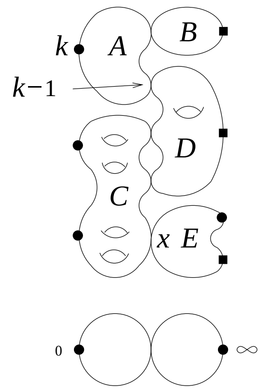


Figure 4: A boundary divisor is isomorphic to a product of several simpler DR-spaces.

curve that have only white preimages (together with these white preimages), (ii) contract the unstable components of the image curve and their preimages in the source curve. If the image curve happens to have only marked points, then we simply ignore the corresponding part. For example, in Figure 4 the part B will be ignored.

Denote by c the number of parts that remain after that ($c = 4$ in our example), and denote by $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_c$ the corresponding DR-spaces. We have constructed a map

$$u : \mathrm{DIV} \to \bar{\mathbf{A}}_1 \times \ldots \times \bar{\mathbf{A}}_c .$$

This map sends the fundamental homology class of DIV to a (multiple of) the fundamental homology class of the product in the image. Indeed, u is an isomorphism on the open part of DIV where ramification points do not coincide.

4. Re-assigning the ψ-classes.

Recall that the map h takes the source curve in Figure 4, forgets all the marked points except the round black dots and stabilizes the curve. We are interested in the preimages of the ψ classes under h. During the stabilization the marked points from certain parts can land on some other parts. In the figure, the ψ-class assigned to the point labeled with k (on the part A) will land on the part D. Similarly, the ψ-class assigned to the round black dot of the part E will land on the part C. Indeed, for *any* admissible covering lying in the boundary divisor represented in the figure, the parts A, B, and E are contracted by h. Once we have re-assigned the ψ-classes in that way, the preimages $h^*(\psi_i)$ on DIV coincide with the classes $h^*(\psi_i)$ defined separately for each part. Indeed, for instance, in our example, stabilizing the source

curve is equivalent to contracting the parts A, B, and E and then stabilizing the parts C and D.

5. Splitting the integral.

Now we can split the integral (4) into a product of similar integrals over $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_c$ with the ψ-classes assigned as explained in paragraph 4.

The total number of ψ-classes in these integrals equals $\sum d_i - 1$, because we replaced one of the ψ-classes by the boundary divisors.

This completes the proof of the proposition. $\diamondsuit$

Proposition 4.1 constitutes the recursive step of our algorithm. To make things precise we must add two comments.

a) The condition $|k_i| > \sum d_i$ is easily seen to be still satisfied for the smaller DR-spaces. Indeed, An index k of a zero or a pole can decrease only by "annihilating" one or several black squares. For instance, in Figure 4 the initial index k has become equal to $k-1$ on the component D by annihilating one square black dots.

b) The restriction of Witten's class (more precisely, of $h^*(c_W)$) on the components is obtained by using the factorization rules. This involves some choices or remainders modulo r. For example, in Figure 4, we will have to sum over $r-1$ possibilities of the remainders at the nodes connecting C to D. The remainders assigned to the sqaure dots are equal to 0 since they are forgotten under the map h.

Proof of Theorem 1. Assume we want to find the value of an integral over $\overline{\mathcal{M}}_{g,n}$ involving Witten's class and powers of the ψ-classes. First, using the above lemmas we can get rid of the ψ-classes and reduce the problem to computing the integral of Witten's class over DR-spaces. Comparing the degree of Witten's class

$$\deg c_W = \frac{(r-2)(g-1) + \sum a_i}{r} \leq \frac{(r-2)(n+g-1)}{r}$$

to the dimension of a DR-cycle

$$\dim = 3g - 3 + n - p \quad (p \leq g),$$

we see that such an integral can be nonzero only in 2 cases: either for genus 0 or for DR-cycles of codimension 1 in genus 1. The genus zero integrals are well-known, while the case of double ramification divisors in genus 1 was treated in Section 3. $\diamondsuit$

Remark 4.2 Our first goal was to compute the integral of $c_W \psi_1^{d_1} \dots \psi_n^{d_n}$ over the moduli spaces $\overline{\mathcal{M}}_{g,n}$, but we actually computed such integrals over all DR-cycles satisfying the strange-looking condition $|k_i| > \sum d_i$. This generalization is unavoidable if we want to make the algorithm work. It is easy to see that even if we start with an integral over a moduli space the integral can immediately lead us to integrals over DR-cycles.

References

[1] **T. J. Jarvis.** *Geometry of the moduli of higher spin curves.* – International Journal of Mathematics, vol. **11** (2000), no. 5, pp. 637–663. arXiv:math.AG/9809138.

[2] **D. Abramovich, T. J. Jarvis.** *Moduli of twisted spin curves.* – Proc. Amer. Math. Soc., vol **131** (2003), no. 3, pp. 685–699 (electronic), arXiv:math.AG/0104154.

[3] **A. Chiodo.** *Witten's top Chern class in K-theory.* – arXiv:math.AG/0210398 (2002), 21 p.

[4] **A. Chiodo.** *Compactifying the stack of higher spin curves without ramifications.* – In preparation. See http://math.unice.fr/~chiodo.

[5] **C. Faber, R. Pandharipande.** *Relative maps and tautological classes.* – J. Eur. Math. Soc. vo. **7**(2005), No. 1, pp. 13–49.

[6] **E.-N. Ionel.** *Topological Recursive Relations in $H^{2g}(\mathcal{M}_{g,n})$.* – Invent. Math., vol **148** (2002), no. 3, pp. 627–658, arXiv:mathAG/9908060.

[7] **M. Kazarian, S. Lando.** *An algebro-geometric proof of Witten's conjecture.* – Max-Planck Institute preprint MPIM2005-55 (2005), http://www.mpim-bonn.mpg.de/preprints, 14 p.

[8] **Y.-S. Kim, K. Liu.** *A simple proof of Witten conjecture through localization.* – arXiv:math.AG/0508384 (2005), 9 p.

[9] **M. Kontsevich.** *Intersection theory on the moduli space of curves and the matrix Airy function.* – Communications in Mathematical Physics, vol. **147** (1992), p. 1–23.

[10] **M. Mirzakhani.** *Weil–Petersson volumes and intersection theory on the moduli space of curves.* – http://abel.math.harvard.edu/~mirzak, 23 p.

[11] **D. Mumford.** *Towards an enumerative geometry of the moduli space of curves.* – In "Arithmetic and Geometry", vol. II, pp. 271–328, Progr. Math., vol. **36**, J. Coates, S. Helgason eds, Birkhäuser Boston, MA, 1983.

[12] **A. Okounkov, R. Pandharipande.** *Gromov-Witten theory, Hurwitz numbers, and Matrix models, I* – `arXiv:math.AG/0101147`, 107 p.

[13] **A. Polishchuk, A. Vaintrob.** *Algebraic construction of Witten's top Chern class.* – Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), pp. 229–249, Contemp. Math., vol **276**, Amer. Math. Soc., Providence, RI, 2001, `arXiv:math.AG/0011032`.

[14] **A. Polishchuk.** *Witten's top Chern class on the moduli space of higher spin curves.* – In "Frobenius manifolds", pp. 253–264, Aspects Math., vol. **E36**, Vieweg, Wiesbaden, 2004, `arXiv:math.AG/0208112`.

[15] **S. Shadrin.** *Geometry of meromorphic functions and intersections on moduli spaces of curves.* – Int. Math. Res. Not. 2003, no. 38, pp. 2051–2094, `arXiv:math.AG/0209282`.

[16] **E. Witten.** *Two dimensional gravity and intersection theory on moduli space.* – Surveys in Differential Geometry, vol. **1** (1991), p. 243–310.

[17] **E. Witten.** *Algebraic geometry associated with matrix models of two dimensional gravity.* – Topological Methods in Modern Mathematics, (Proceedings of Stony Brook, NY, 1991), Publish or Perish, Houston, 1993, p. 235–269.